APPLICATIONS OF NONVARIATIONAL FINITE ELEMENT METHODS TO MONGE–AMPÈRE TYPE EQUATIONS

TRISTAN PRYER

Abstract. The goal of this work is to illustrate the application of the nonvariational finite element method to a specific Monge-Ampère type nonlinear partial differential equation. The equation we consider is that of prescribed Gauss curvature although the method can be generalised to any Monge-Ampère operator.

1. Introduction and problem setting

The *nonvariational finite element method* (NVFEM) introduced in [LP11a] is a numerical method aimed at problems of the form

$$\boldsymbol{A}(\boldsymbol{x}){:}\mathrm{D}^2 u(\boldsymbol{x}) = f(\boldsymbol{x}) \tag{1.1}$$

where for each $\boldsymbol{x} \in \Omega$ the matrix $\boldsymbol{A}(\boldsymbol{x}) \in \mathrm{Sym}(\mathbb{R}^{d\times d})$, the space of bounded symmetric positive definite matrices and where $\mathrm{D}^2 u$ denotes the Hessian of the function u. The operation $\boldsymbol{B}{:}\boldsymbol{C} = \mathrm{trace}\,(\boldsymbol{B}^\intercal \boldsymbol{C})$ is the Frobenius inner product between two $d \times d$ matrices. Classical finite element methods are applicable to this problem if we assume the coefficient matrix $\boldsymbol{A}$ is differentiable. In this case we may rewrite (1.1) in *variational* or *divergence* form via the introduction of an advection term since

$$f(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x}){:}\mathrm{D}^2 u(\boldsymbol{x}) = \mathrm{div}\,((\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x}))) - \mathrm{D}\boldsymbol{A}(\boldsymbol{x})\nabla u(\boldsymbol{x}) \tag{1.2}$$

where

$$\mathrm{D}\boldsymbol{A}(\boldsymbol{x}) = \left(\sum_{i=1}^{d} \partial_i a_{i,1}(\boldsymbol{x}), \dots, \sum_{i=1}^{d} \partial_i a_{i,d}(\boldsymbol{x})\right). \tag{1.3}$$

Note that we are using the convention that $\nabla\phi = (\partial_1\phi, \dots, \partial_d\phi)^\intercal$ is the column vector formed of first order partial derivatives of a d–multivariate function ϕ.

The introduction of the advection term may result in the variational problem becoming advection dominated. This is undesirable in the finite element context and stabilisation terms become necessary to derive a viable numerical method [EG04, c.f.]. Interestingly if $\|\mathrm{D}\boldsymbol{A}\|_{\mathrm{L}_\infty(\Omega)} \gg \|\boldsymbol{A}\|_{\mathrm{L}_\infty(\Omega)}$ applying the NVFEM to (1.1) does not result in an unstable scheme, whereas applying a standard FEM to (1.2) does. This is numerically demonstrated in [LP11a, §4.2]. It may even be the case that $\boldsymbol{A}$ is not differentiable, in which case the standard FEM cannot be applied.

The fully nonlinear problem

$$\mathscr{F}(\mathrm{D}^2 u) = 0 \tag{1.4}$$

is related to the nonvariational problem (1.1) by the fundamental theorem of calculus. If $\mathscr{F}$ is sufficiently regular and u solves (1.4) then u also solves

$$\left[\int_0^1 \mathscr{F}'(t\mathrm{D}^2 u)\,\mathrm{d}t\right]{:}\mathrm{D}^2 u + \mathscr{F}(0) = 0, \tag{1.5}$$

Date: February 24, 2013.

where $\mathscr{F}'$ is the Fréchet derivative of $\mathscr{F}$. It is also the case that a Newton linearisation applied to (1.4) results in a sequence of linear nonvariational PDEs:

$$\mathscr{F}'(\mathrm{D}^2 u^n):\mathrm{D}^2\left(u^{n+1}-u^n\right) = -\mathscr{F}(\mathrm{D}^2 u^n). \tag{1.6}$$

The Monge-Ampère operators are an extremely interesting class of fully nonlinear PDE. These arise from differential geometry and optimal transport problems; they take the form

$$\mathscr{F}(\mathrm{D}^2 u) := \det\left(\mathrm{D}^2 u\right) - f\left(\nabla u, u, \boldsymbol{x}\right) = 0. \tag{1.7}$$

For example the Monge–Ampère–Dirichlet (MAD) problem is the case when $f = f(\boldsymbol{x})$ and (1.7) is coupled with a Dirichlet type boundary condition ($u = g$ on $\partial\Omega$). This particular equation is a prototypical example of a fully nonlinear PDE.

There are a variety of numerical methods available for the more general Monge-Ampère class of fully nonlinear PDE (1.7). In [Obe08] the author proposes a wide stencil finite difference scheme. In [Böh08] a C^1 finite element scheme based on the Argyris element is used. In a series of papers Feng and Neilan [FN09b, FN09a] construct numerical approximations of solutions to sequences of quasilinear biharmonic equations. This is very reminiscent of the vanishing viscosity method first studied for use in fully nonlinear first order PDEs. The method is aptly named the vanishing moment method. More recently in [BGNS11] a consistent penalisation method has been introduced for these problems. Finally, Awanou [Awa11] uses a *Laplacian relaxation* technique to study these equations.

For the Monge-Ampère type equation (1.7) to be well posed we require $\Omega \subset \mathbb{R}^d$ to be a convex domain and $f > 0$. The Monge-Ampère operator is elliptic over the cone of strictly convex functions in Ω and under the constraints above will admit a unique convex viscosity solution [CC95].

In this work we will study the equation of prescribed Gauss curvature. This arises from the problem of finding a function u such that the graph of u has a specified Gaussian curvature K. In this case we have that

$$K = \frac{\det \mathrm{D}^2 u}{\left(1 + |\nabla u|^2\right)^{(d+2)/2}} \tag{1.8}$$

and hence

$$\mathscr{F}(\mathrm{D}^2 u, \nabla u, \boldsymbol{x}) := \det \mathrm{D}^2 u - K\left(1 + |\nabla u|^2\right)^{(d+2)/2}. \tag{1.9}$$

Note that $K = K(u, \boldsymbol{x})$.

The linearisation of this problem can be calculated in a direction v as

$$\begin{aligned}\mathscr{F}'(\mathrm{D}^2 u, \nabla u, \boldsymbol{x}):\mathrm{D}^2 v &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathscr{F}(\mathrm{D}^2 u + \epsilon \mathrm{D}^2 v, \nabla u + \epsilon\nabla v, \boldsymbol{x}) - \mathscr{F}(\mathrm{D}^2 u, \nabla u, \boldsymbol{x})\right)\\ &= \operatorname{cof}\mathrm{D}^2 u:\mathrm{D}^2 v + (d+2)\,K\left(\left(1 + |\nabla u|^2\right)^{d/2}(\nabla u)^{\intercal}\nabla v\right)\end{aligned} \tag{1.10}$$

and thus the linearisation is elliptic if $\operatorname{cof}\mathrm{D}^2 u$ is an elliptic operator. This holds for convex u.

2. Discretisation

The process of discretisation can be sought in two ways. We may look at the continuous problem and discretise this directly, resulting in a system of nonlinear equations, or we may first linearise the problem and discretise from there. Discretising the nonlinear problem directly is certainly possible but is more technical, as discussed in [BGNS11]. For brevity we will perform a Newton linearisation on

(1.7) and discretise the sequence of linear nonvariational PDEs in a similar light to [LP11b].

Let $\mathscr{T}$ be a conforming, shape regular triangulation of Ω, namely, $\mathscr{T}$ is a finite family of sets such that

(1) $K \in \mathscr{T}$ implies K is an open simplex (segment for $d = 1$, triangle for $d = 2$, tetrahedron for $d = 3$),
(2) for any $K, J \in \mathscr{T}$ we have that $\overline{K} \cap \overline{J}$ is a full sub-simplex (i.e., it is either $\emptyset$, a vertex, an edge, a face, or the whole of $\overline{K}$ and $\overline{J}$) of both $\overline{K}$ and $\overline{J}$ and
(3) $\bigcup_{K \in \mathscr{T}} \overline{K} = \overline{\Omega}$.

We also define $\mathscr{E}$ to be the skeleton of the triangulation, that is the set of sub-simplices of $\mathscr{T}$ contained in Ω but not $\partial\Omega$. For $d = 2$, for example, $\mathscr{E}$ would consist of the set of edges of $\mathscr{T}$ not on the boundary.

We use the convention where $h : \Omega \to \mathbb{R}$ denotes the *meshsize function* of $\mathscr{T}$, i.e.,

$$h(\boldsymbol{x}) := \max_{\overline{K} \ni \boldsymbol{x}} h_K. \tag{2.1}$$

2.1. **Definition** (FE spaces). Let $\mathbb{P}^p(\mathscr{T})$ denote the space of piecewise polynomials of degree k over the triangulation $\mathscr{T}$ of Ω. We introduce the *finite element spaces*

$$\mathbb{V} = \mathbb{P}^p(\mathscr{T}) \cap \mathrm{C}^0(\Omega) \cap \mathrm{H}^1_0(\Omega) \text{ and } \mathbb{W} = \mathbb{P}^p(\mathscr{T}) \cap \mathrm{C}^0(\Omega) \tag{2.2}$$

to be the usual space of continuous piecewise polynomial functions and

$$\mathbb{S} := \mathbb{V} \times \mathbb{W}^{d^2}. \tag{2.3}$$

2.2. **Remark** (generalised Hessian). Given a function $v \in \mathrm{H}^2(\Omega)$, let $\boldsymbol{n} : \partial\Omega \to \mathbb{R}^d$ be the outward pointing normal of Ω then the Hessian of v, $\mathrm{D}^2 v$, satisfies the following identity:

$$\left\langle \mathrm{D}^2 v \,\middle|\, \phi \right\rangle = -\int_\Omega \nabla v \otimes \nabla\phi + \int_{\partial\Omega} \nabla v \otimes \boldsymbol{n}\ \phi \quad \forall\, \phi \in \mathrm{H}^1(\Omega) \tag{2.4}$$

where $\langle \cdot \,|\, \cdot \rangle$ denotes an appropriate duality pairing. It follows that we may weaken the regularity assumptions to $v \in \mathrm{H}^1(\Omega) \cap \mathrm{H}^1(\partial\Omega)$.

2.3. **Definition** (finite element Hessian). From Remark 2.2 and in view of Reisz representation theorem we may define the *finite element Hessian* such that

$$\int_\Omega \boldsymbol{H}[V]\Phi = \left\langle \mathrm{D}^2 V \,\middle|\, \Phi \right\rangle \quad \forall\, \Phi \in \mathbb{W}. \tag{2.5}$$

2.4. **Proposition** (symmetry of the finite element Hessian). *The finite element Hessian is symmetric, that is for each $V \in \mathbb{V}$*

$$\int_\Omega \boldsymbol{H}[V]\Phi = \int_\Omega (\boldsymbol{H}[V])^\intercal \Phi \quad \forall\, \Phi \in \mathbb{W}. \tag{2.6}$$

In view of the constraints to the continuous problem (1.9) to admit a unique solution it is also necessary to construct a discrete notion of convexity. This has been developed in [AM09] and is naturally passed down from the concept of distributional convexity.

2.5. **Definition** (finite element convexity [AM09]). A function, $v \in \mathrm{H}^1(\Omega) \cap \mathrm{H}^1(\partial\Omega)$, is said to be *finite element convex* if

$$\int_\Omega \boldsymbol{H}[v]\Phi \text{ is positive semidefinite} \quad \forall\, \Phi \in \mathbb{W} \tag{2.7}$$

where $\Phi \geq 0$ on Ω. It is strictly finite element convex if (2.7) is positive definite.

Definition 2.3 allows us to construct what is essentially a 2–0 mixed method where the Hessian of the solution to (1.7) is treated as an auxiliary variable in the formulation (as opposed to the 1–1 mixed methods commonly found in the literature by decoupling a second order PDE into a system of first order PDEs [BF91, c.f.]). Given the linearisation (1.10) we formulate the problem in the discrete setting as follows: Given an initial guess $\left(U^0, \boldsymbol{H}[U^0]\right) \in \mathbb{S}$ that is strictly finite element convex (2.7), for $n \in \mathbb{N}$ find $(U^n, \boldsymbol{H}[U^n]) \in \mathbb{S}$ such that

$$\langle \boldsymbol{H}[U^n], \Phi\rangle + \langle \nabla U^n \otimes \nabla \Phi\rangle - \sum_{e\in\partial\Omega} \langle \nabla U^n \otimes \boldsymbol{n}\ \Phi\rangle_e = 0 \tag{2.8}$$

$$\left\langle \mathscr{F}'\left(\boldsymbol{H}[U^{n-1}], \nabla U^{n-1}\right):\boldsymbol{H}[U^n - U^{n-1}] + \mathscr{F}\left(\boldsymbol{H}[U^{n-1}], \nabla U^{n-1}\right), \Phi\right\rangle = 0 \quad \forall\, \Phi \in \mathbb{V}. \tag{2.9}$$

Where for the problem of prescribed Gaussian curvature (2.9) is

$$\begin{aligned}\left\langle \mathscr{F}'\left(\boldsymbol{H}[U^{n-1}], \nabla U^{n-1}\right):\boldsymbol{H}[U^n - U^{n-1}] + \mathscr{F}\left(\boldsymbol{H}[U^{n-1}], \nabla U^{n-1}\right), \Phi\right\rangle \\ = \int_\Omega \operatorname{cof} \boldsymbol{H}[U^{n-1}]:\boldsymbol{H}[U^n - U^{n-1}]\Phi \qquad\qquad (2.10)\\ + \int_\Omega 2dK\left(1 + \left|\nabla U^{n-1}\right|^2\right)^{d/2} \left(\nabla U^{n-1}\right)^{\intercal}\nabla\left(U^n - U^{n-1}\right)\Phi \\ + \int_\Omega \left(\det \boldsymbol{H}[U^{n-1}] - K\left(1 + \left|\nabla U^{n-1}\right|^2\right)^{(d+2)/2}\right)\Phi.\end{aligned}$$

Due to the symmetry property given in Proposition 2.4 we may simplify the problem somewhat to seeking only the upper (or lower) triangular parts of the finite element Hessian. This reduces $\mathbb{S} = \mathbb{V} \times \mathbb{W}^{(d^2+d)/2}$.

2.6. **Theorem** (solvability of the discrete system [Pry10])**.** *Let $U \in \mathbb{V}$ be the nonvariational finite element approximation to u, the solution of*

$$\boldsymbol{A}:\mathrm{D}^2 u = f, \tag{2.11}$$

where $\boldsymbol{A}$ is an elliptic operator. Then we have a discrete inf–sup condition, that is the linear system is always invertible. Hence, assuming the linearisation maintains ellipticity, the discrete problem (2.8)– (2.9) is well posed.

3. Numerical experiments

In this section we detail numerical experiments on the formulation (2.8)– (2.9).

We will consider the case $d = 2$ and when $K > 0$ is some prescribed curvature. In each of the experiments we choose $p = 2$, i.e., $\mathbb{V}$ consists of piecewise quadratic functions. The domain Ω is taken as a square whose size differs on each of the experiments and the triangulation $\mathscr{T}$ is unstructured. All of the numerical experiments have been conducted using the DOLFIN environment of the finite element package FEniCS [LW10].

In Figures 1–2 we construct classical solutions to (1.9) in order to look at the numerical convergence of the method. In Figure 3 we consider K as a constant over the domain $[-.57, .57]^2$. These results can then be compared with the two other numerical studies found in the literature on prescribed Gauss curvature [FN09b, Awa11]. In these experiments the authors note that the problem (1.9) is well posed only for $K \leq K^{\max}$ and estimate the value of $K^{\max}$ by asserting when the numerical algorithm proposed breaks down.

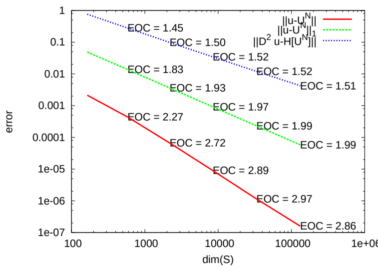
The initial guess to any Newton iteration is paramount due to the well known *overshoot* property. In the case of Monge-Ampère type linearisations it's especially important since (discrete) convexity must be maintained during the iterative procedure for the problem to remain well posed. In each of the tests below we initialise

the algorithm by approximating the solution of the MAD problem over the initial mesh as detailed in [LP11b].

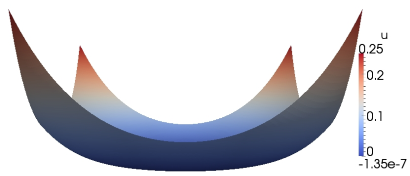
$$\begin{aligned} \det \mathrm{D}^2 u &= K \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{3.1}$$

The iterative procedure given by the discrete problem (2.8)– (2.9) is terminated when two concurrent iterates satisfy $\left\|U^n - U^{n-1}\right\|_{\mathrm{L}_\infty(\Omega)} \leq 10^{-10}$.

FIGURE 1. In this experiment we fix choose a convex solution u which classically solves the equation of prescribed Gauss curvature (1.9) over the square $[-.5, .5]^2$. That is, we fix $u = |\boldsymbol{x}|^4$ and calculate $K = K(\boldsymbol{x}, u)$. We solve the discrete problem over a sequence of concurrently refined meshes and ascertain the errors and convergence rates for the problem in $\mathrm{L}_2(\Omega)$, $\mathrm{H}^1_0(\Omega)$ and a discrete $\mathrm{H}^2(\Omega)$ seminorm. Notice that $\left\|u - U^N\right\| \approx \mathrm{O}(h^3)$, $\left|u - U^N\right|_1 \approx \mathrm{O}(h^2)$ and $\left\|\mathrm{D}^2 u - \boldsymbol{H}[U^N]\right\| \approx \mathrm{O}(h^{1.5})$.



(a) Errors and convergence rates for the problem, $p = 2$.



(b) Solution plot

REFERENCES

[AM09] Néstor E. Aguilera and Pedro Morin. On convex functions and the finite element method. *SIAM J. Numer. Anal.*, 47(4):3139–3157, 2009.

[Awa11] Gerard. Awanou. Pseudo time continuation and time marching methods for monge–ampere type equations. *Preprint www.math.niu.edu/ awanou/*, 2011.

[BF91] Franco Brezzi and Michel Fortin. *Mixed and hybrid finite element methods*, volume 15 of *Springer Series in Computational Mathematics*. Springer-Verlag, New York, 1991.

[BGNS11] Susanne C. Brenner, Thirupathi Gudi, Michael Neilan, and Li-yeng Sung. C^0 penalty methods for the fully nonlinear Monge-Ampère equation. *Math. Comp.*, 80(276):1979–1995, 2011.

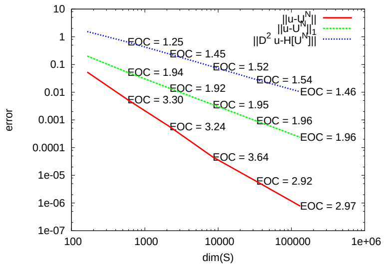
[Böh08] Klaus Böhmer. On finite element methods for fully nonlinear elliptic equations of second order. *SIAM J. Numer. Anal.*, 46(3):1212–1249, 2008.

[CC95] Luis A. Caffarelli and Xavier Cabré. *Fully nonlinear elliptic equations*, volume 43 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1995.

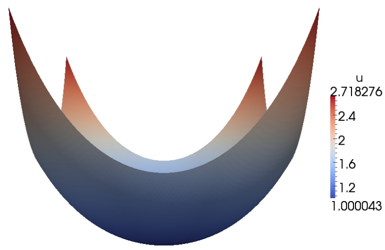
[EG04] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[FN09a] Xiaobing Feng and Michael Neilan. Mixed finite element methods for the fully nonlinear Monge-Ampère equation based on the vanishing moment method. *SIAM J. Numer. Anal.*, 47(2):1226–1250, 2009.

FIGURE 2. In this experiment we fix choose a convex solution u which classically solves the equation of prescribed Gauss curvature (1.9) over the square $[-.5, .5]^2$. That is, we fix $u = \exp\left(|\boldsymbol{x}|^2/2\right)$ and calculate $K = K(\boldsymbol{x}, u)$. We solve the discrete problem over a sequence of concurrently refined meshes and ascertain the errors and convergence rates for the problem in $\mathrm{L}_2(\Omega)$, $\mathrm{H}^1_0(\Omega)$ and a discrete $\mathrm{H}^2(\Omega)$ seminorm. Notice that $\left\|u - U^N\right\| \approx \mathrm{O}(h^3)$, $\left|u - U^N\right|_1 \approx \mathrm{O}(h^2)$ and $\left\|\mathrm{D}^2 u - \boldsymbol{H}[U^N]\right\| \approx \mathrm{O}(h^{1.5})$.



(a) Errors and convergence rates for the problem, $p = 2$.



(b) Solution plot

[FN09b] Xiaobing Feng and Michael Neilan. Vanishing moment method and moment solutions for fully nonlinear second order partial differential equations. *J. Sci. Comput.*, 38(1):74–98, 2009.

[LP11a] Omar Lakkis and Tristan Pryer. A finite element method for second order nonvariational elliptic problems. *SIAM J. Sci. Comput.*, 33(2):786–801, 2011.

[LP11b] Omar Lakkis and Tristan Pryer. A nonvariational finite element method for nonlinear elliptic problems. *Submitted - tech report available on ArXiV http://arxiv.org/abs/1103.2970*, 2011.

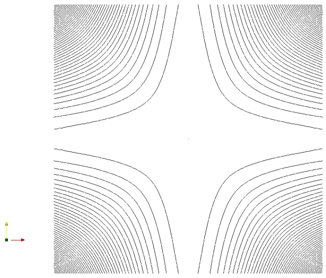
[LW10] Anders Logg and Garth N. Wells. DOLFIN: automated finite element computing. *ACM Trans. Math. Software*, 37(2):Art. 20, 28, 2010.

[Obe08] Adam M. Oberman. Wide stencil finite difference schemes for the elliptic Monge-Ampère equation and functions of the eigenvalues of the Hessian. *Discrete Contin. Dyn. Syst. Ser. B*, 10(1):221–238, 2008.

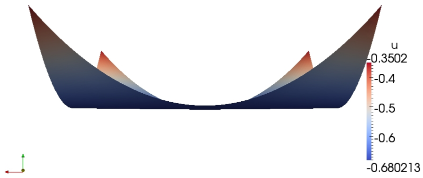
[Pry10] Tristan Pryer. Recovery methods for evolution and nonlinear problems. *DPhil Thesis, University of Sussex*, 2010.

TRISTAN PRYER
DEPARTMENT OF MATHEMATICS, STATISTICS AND ACTUARIAL SCIENCE
UNIVERSITY OF KENT
CANTERBURY
CT2 7NF, UNITED KINGDOM
E-mail address: T.Pryer@kent.ac.uk

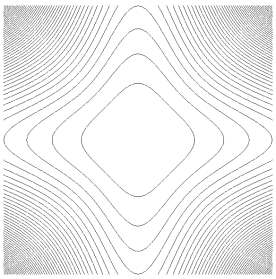
FIGURE 3. In this experiment we fix $h \approx 0.009$ and implement the discrete problem over an unstructured mesh of the square $[-0.57, 0.57]^2$ consider the case K is constant. We choose various values of $K > 0$ and display a contour plot together with a side view of the discrete solution. Note that the numerical algorithm fails to converge for $K = 2$.



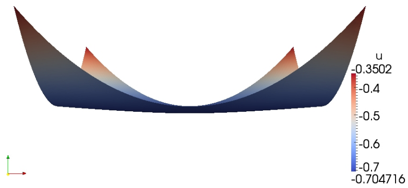
(a) Contour plot for $K = 0.01$



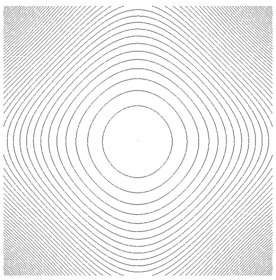
(b) Solution plot for $K = 0.01$



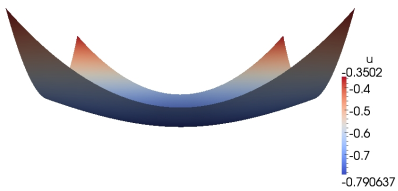
(c) Contour plot for $K = 0.1$



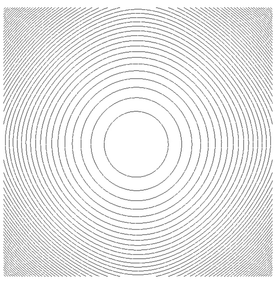
(d) Solution plot for $K = 0.1$



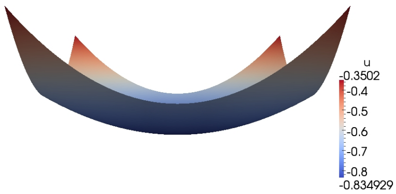
(e) Contour plot for $K = 0.5$



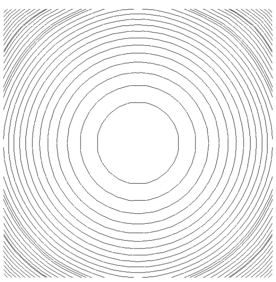
(f) Solution plot for $K = 0.5$



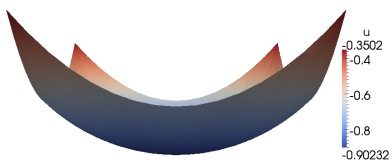
(g) Contour plot for $K = 1.0$



(h) Solution plot for $K = 1.0$



(i) Contour plot for $K = 1.5$



(j) Solution plot for $K = 1.5$